Double Vizing fans in critical class two graphs

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Abstract

Let G be a simple graph and $\chi'(G)$ be the chromatic index of G. We call G a Δ-critical graph if $\chi'(G) = \Delta(G) + 1$ and $\chi'(H) \le \Delta(G)$ for every proper subgraph H of G, where $\Delta(G)$ is the maximum degree of G. Let $e = xy$ be an edge of G and φ be an edge $\Delta(G)$-coloring of $G - e$. An *e-fan* is a sequence $F^e = (x, e, y, e_1, z_1, \dots, e_p, z_p)$ of alternating vertices and distinct edges such that edge e_i is incident with x or y, z_i is another endvertex of e_i and $\varphi(e_i)$ is missing at a vertex before z_i for each i with $1 \le i \le p$. We prove that if $\min\{d(x), d(y)\} \le \Delta(G) - 1$, where $d(x)$ and $d(y)$ respectively denote the degrees of vertices x and y, then colors missing at different vertices of $V(F^e)$ are distinct. Clearly, a Vizing fan is an e-fan with the restricting that all edges e_i being incident with one fixed endvertex of edge e.

This result gives a common generalization of several recently developed new results on multi-fan, double fan, Kierstead path of four vertices, and broom. By treating some colors of edges incident with vertices of low degrees as missing colors, Kostochka and Stiebitz introduced C-fan. We also generalize C-fans from centered at one vertex to one edge.

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1 Introduction

Our results in this paper are on simple graphs, but we will mention some definitions and results on multigraphs. Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a (multi)-graph G, respectively; and by $[k]$ the set of first k consecutive positive integers. We will generally follow the book [**?**] of Stiebitz et al. for notation and terminology. The *edge k-coloring* of a (multi)-graph G is a mapping from $E(G)$ to $[k]$ such that distinct adjacent edges have different values. Denote by $\mathcal{C}^k(G)$ the set of all edge k-colorings of G. The minimum number k, denoted by $\chi'(G)$, such that $\mathcal{C}^k(G) \neq \emptyset$ is called the *chromatic index* of G. An edge e is *critical* if $\chi'(G-e) = \chi'(G) - 1$; and a (multi)-graph G is *k-critical* if $\chi'(G) = k+1$ and $\chi'(H) \leq k$ for every proper subgraph H of G. Vizing [**?**, **?**] and Gupta [**?**] independently proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$, where $\Delta(G)$ and $\mu(G)$ are maximum degree and multiplicity of G, respectively. When G is a simple graph, we have $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$, and so simple graphs are divided into two families: class one and class two accordingly. A critical class two graph is called a *Δ-critical graph.*

Let e be an edge of a k-critical (multi)-graph G and $\varphi \in \mathcal{C}^k(G-e)$. For a vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors assigned to edges incident with v, and $\overline{\varphi}(v) = [k] - \varphi(v)$, i.e., the set of colors not assigned to any edge incident with v. We call $\varphi(v)$ the set of colors *present* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. Clearly, $|\varphi(v)|+|\overline{\varphi}(v)| = k$ for each vertex $v \in V(G)$. A vertex set $X \subseteq V(G)$ is *φ-elementary*, or simply *elementary*, if $\overline{\varphi}(x) \cap \overline{\varphi}(y) = \emptyset$ for every pair of two distinct vertices x, $y \in X$. If $V(G)$ is φ-elementary, then each color is missing at exact one vertex of G. So $n = |V(G)|$ is odd and $|E(G)| = \frac{1}{2}k(n-1)+1$; and G is called *overfull* in this case. Recently, Chen et al. [**?**] proved the Goldberg-Seymour conjecture that if G is a k-critical multigraph with $k \geq \Delta(G)+1$, then for every edge e there exists a coloring $\varphi \in \mathcal{C}^k(G-e)$ such that $V(G)$ is φ-elementary, which is equivalent to that G is overfull. Consequently, $V(G)$ is elementary for every edge k-coloring of $G-e$. This result gives a complete characterization for critical multigraphs G with chromatic index at least $\Delta(G)+2$. However, characterizing elementary sets for k-critical graphs with $k = \Delta(G)$, in particular, Δ-critical simple graphs, is an interesting yet challenging problem in graph edge coloring.

Let G be a Δ-critical graph, $e \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. We in general do not know much about the largest φ-elementary sets except the following three outstanding conjectures. Hilton's overfull conjecture [**?**, **?**]: $V(G)$ is φ-elementary if $\Delta(G) > |V(G)|/3$; Seymour's exact conjecture [**?**]: $V(G)$ is φ-elementary if G is a planar graph; and Hilton and Zhao's core conjecture [**?**]: $V(G)$ is φ-elementary if the core G_Δ has maximum degree at most 2, where G_Δ, named the *core* of G, is the subgraph of G induced by all maximum degree vertices. Cao et al. [**?**] recently confirmed Hilton and Zhao's core conjecture. The other two of these three conjectures are remaining wild open. Vizing [**?**, **?**] showed that the vertex set of every Vizing fan is elementary. Almost all known techniques in studying edge chromatic problems are built on the elementary properties of Vizing fans and its generalizations. In [**?**], Stiebitz et al. gave a survey, up to that time, of the work in this direction. We will give a common generalization of these results. For $x, y \in V(G)$, let $E_G(x, y)$ denote the set of all edges of G joining vertices x and y.

Definition 1.1 (Tashkinov Tree)**.** Let G be a k-critical (multi)-graph, $e \in E_G(x, y)$ and $\varphi \in \mathcal{C}^k(G-e)$ for some integer $k \geq 0$. A sequence $T = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating distinct vertices and distinct edges is called a *Tashkinov tree* if for each $i \in [p]$, e_i is incident with z_i and satisfies the following two conditions.

T1. The other endvertex of e_i is in $\{x, y, z_1, \ldots, z_{i-1}\}$.

T2. $\varphi(e_i) \in \overline{\varphi}(x) \cup \overline{\varphi}(y) \cup \overline{\varphi}(z_h)$ for some $h \in [i-1]$.

Tashkinov trees are given by Tashkinov in [**?**], where he proved that if G is a k-critical multigraph with $k \geq \Delta(G) + 1$, $e \in E(G)$ and $\varphi \in \mathcal{C}^k(G - e)$, then the vertex set of every Tashkinov tree is φ-elementary. Clearly, each Tashkinov tree is indeed a tree. We in the following notice that Vizing fans and some other well-studied subgraphs are special classes of Tashkinov trees.

1. If we restrict in **T1**, each e_i is incident with x and in **T2** $h = i - 1$, then T is a *Vizing fan*.
2. If we only impose the above restriction to **T1**, then T is a *multi-fan* introduced by Stiebitz et al. [**?**].
3. If we restrict in **T1**, e_1 is incident with y and e_i is incident with z_{i-1} for each $i \geq 2$, then T is a *Kierstead path* [**?**].
4. If we restrict in **T1**, $p \geq 2$ and each e_i with $i \geq 2$ is incident with z_1, then T is a *broom* defined in [**?**, **?**].

We notice that not every vertex set of Tashkinov tree is elementary. Let P^* be obtained from the Petersen graph by deleting a vertex. It is not difficult to verify that P^* is a 3-critical graph, but there exist an edge e and a coloring $\varphi \in \mathcal{C}^3(P^* - e)$, such that the vertex set of a Kierstead path with 4 vertices is not elementary. For $u \in V(G)$, let $d(u)$ denote the *degree* of vertex u in G. By imposing degree condition $\min\{d(y), d(z_1)\} \leq \Delta(G) - 1$, Stiebitz and Kostachka [**?**] and Luo and Zhao [**?**] showed that the vertex set of each Kierstead path $(x, e, y, e_1, z_1, e_2, z_2)$ is elementary. The result has been extended to brooms [**?**, **?**]. We generalize these results to a much broader class of Tashkinov trees in this paper.

Definition 1.2 (e-fan)**.** Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. A Tashkinov tree $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ is a *simple e-fan* if in **T1** we additionally require each e_i is only incident with x or y i.e., $e_i = xz_i$ or $e_i = yz_i$. Furthermore, in the above definition of simple e-fan if we relax the condition that each z_i is distinct by allowing it with possibility to be repeated one more time, say $z_i = z_j = z$ with $i \neq j$, i.e., edges xz and yz can appear in F^e, then F^e is called an *e-fan*.

Clearly, a multi-fan is an e-fan in simple graphs. Moreover, if F_x and F_y are two multi-fans centered at x and y, respectively, then $F_x \cup F_y$, named a *double fan*, is also an e-fan. The below Theorem 1.3 shows that the vertex set of every e-fan provided $\min\{d(x), d(y)\} \leq \Delta(G) - 1$ is elementary, which is one of the two main results of this paper. We will give its proof in Section 4, in which it is worth mentioning that we first prove the vertex set of some special subsequence (will be called linear e-sequence) is elementary, then generalize to any two special subsequences and finally to the entire e-fan. Actually, a Vizing fan is such a special subsequence centered at one vertex in a multi-fan, so one can also use our above method to prove the vertex set of every multi-fan is elementary.

Theorem 1.3. *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. If $\min\{d(x), d(y)\} \leq \Delta(G) - 1$, then $V(F^e)$ is φ-elementary for every e-fan F^e. Furthermore, if F^e is maximal, i.e., there is no e-fan containing F^e as a proper subsequence, then*

$$d(x) + d(y) - 2\Delta + \sum_{z \in V(F^e)\setminus\{x,y\}} (2d(z) + \mu_{F^e}(x, z) + \mu_{F^e}(y, z) - 2\Delta) = 2,$$

where $\mu_{F^e}(x, z)$ and $\mu_{F^e}(y, z)$ taking value 0 or 1 are the number of edges between x and z and between y and z in F^e, respectively.

We notice that Theorem 1.3 immediately gives that all vertex sets of Vizing fans, multi-fans, and double fans provided $\min\{d(x), d(y)\} \leq \Delta(G) - 1$ are respectively elementary. We also notice a few applications below.

Corollary 1.4 (Kostachka and Stiebitz [**?**], and Luo and Zhao [**?**])**.** *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. For any Kierstead path $K = (x, e, y, e_1, z_1, e_2, z_2)$, if $\min\{d(y), d(z_1)\} \leq \Delta(G) - 1$, then $V(K)$ is φ-elementary.*

Proof. Let φ' be obtained from $\varphi \in \mathcal{C}^\Delta(G-e)$ by uncoloring e_1 and coloring e with color $\varphi(e_1)$. Since $\varphi(e_1) \in \overline{\varphi}(x)$, φ' is an edge $\Delta(G)$-coloring of $G - e_1$. Moreover, since $\varphi'(e) \in \overline{\varphi}(z_1)$ and $\varphi'(e_2) \in \overline{\varphi}'(x) \cup \overline{\varphi}'(y)$, $F^e = (y, e_1, z_1, e, x, e_2, z_2)$ is an e-fan with respect to e_1 and φ'. By Theorem 1.3, $V(F^e) = V(K)$ is φ'-elementary, and so φ-elementary. □

Using the same trick in the above proof, we get the following more general result.

Corollary 1.5 (Cao, Chen, Jing, Stiebitz and Toft [**?**])**.** *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. For any broom $B = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $\min\{d(y), d(z_1)\} \leq \Delta(G) - 1$, then $V(B)$ is φ-elementary.*

2 Adding colors of edges incident with vertices with small degrees to missing color sets

In this section we will consider some extensions of the missing color set at a vertex and some more generally elementary properties and structures. Starting with Vizing's classic results [?, ?], missing colors have played a crucial role in revealing properties of critical graphs. Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$. Woodall [?, ?] treated colors $\varphi(yz)$ as a missing color in $\overline{\varphi}(y)$ if $d(z)$ is "small". This technique was used in [?, ?, ?, ?] in their work on Vizing's average degree conjecture and hamiltonian property of Δ-critical graphs. For a vertex $v \in V(G)$, let

$$\begin{aligned} \varphi^s_x(v) &= \{\varphi(vw) : w \neq x \text{ and } d(w) \leq \frac{1}{2}(\Delta(G) - d(x))\} \text{ and },\\ C_{\varphi,x}(v) &= \overline{\varphi}(v) \cup \varphi^s_x(v). \end{aligned}$$

Similarly, we define $\varphi^s_y(v)$ and $C_{\varphi,y}(v)$. Since $d(x) + d(w) \geq \Delta(G) + 2$ for every neighbor w of x [?], we have $\varphi^s_x(x) = \emptyset$, i.e., $C_{\varphi,x}(x) = \overline{\varphi}(x)$. Similarly, $\varphi^s_y(y) = \emptyset$, i.e., $C_{\varphi,y}(y) = \overline{\varphi}(y)$. Incorporating this idea, Kostochka and Stiebitz [?] extended multi-fan as follows. A sequence $F^c = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating distinct vertices and distinct edges is called a *C-fan* if $V(F^c)$ induces a star centered at x, and for each $i \geq 1$, there exists a h with $0 \leq h \leq i - 1$ such that $\varphi(e_i) \in C_{\varphi,x}(z_h)$, where $z_0 = y$. The set $V(F^c)$ is called *φ^c-elementary* if $C_{\varphi,x}(z_i) \cap C_{\varphi,x}(z_j) = \emptyset$ for every two distinct vertices z_i, z_j in $V(F^c)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$.

Theorem 2.1 (Kostochka and Stiebitz [?]). *Let G be Δ-critical graph, $e \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$. Then $V(F^c)$ is φ^c-elementary for every C-fan F^c.*

Definition 2.2 (C-e-fan). Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$. A *$C$-$e$-fan* at x and y is a sequence $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating vertices and edges satisfying the following two conditions.

C1. The edges $e, e_1, \ldots, e_p$ are distinct with $e_i = xz_i$ or $e_i = yz_i$.

C2. $\varphi(e_i) \in C_{\varphi,y}(x) \cup C_{\varphi,x}(y) \cup C_{\varphi,w(e_h)}(z_h)$ for some $h \in [i-1]$, where $w(e_h)$ is the endvertex of e_h in $\{x, y\}$.

Since each edge e_i with $i \in [p]$ is incident with x or y, let $w(e_i)$ denote this vertex. Note that some vertices of $z_1, \ldots, z_p$ may appear twice, say $z_i = z_j = z$ with $i \neq j$, i.e., edges xz and yz appear in F^{ce}. In C-e-fan F^{ce}, we define $C_\varphi(x) = C_{\varphi,y}(x)$, $C_\varphi(y) = C_{\varphi,x}(y)$, $C_\varphi(z_i) = C_{\varphi,w(e_i)}(z_i)$ for single z_i, and $C_\varphi(z) = C_{\varphi,w(e_i)}(z_i) \cup C_{\varphi,w(e_j)}(z_j)$ for repeated z_i and z_j with $z_i = z_j = z$. The set $V(F^{ce})$ is called *φ^{ce}-elementary* if $C_\varphi(u) \cap C_\varphi(v) = \emptyset$ for every two distinct vertices u, v in $V(F^{ce})$. The below Theorem 2.3 is the other of the two main results of this paper, whose proof will be given in Section 5 and has the similar main idea of Theorem 1.3 but much more complicated.

Theorem 2.3. *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. For a C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \dots, e_p, z_p)$, if $\max\{d(x), d(y)\} \leq \Delta(G) - 1$ and the following condition holds, then $V(F^{ce})$ is φ^{ce}-elementary.*

C3. *For any two distinct colors α, β with $\alpha \in \varphi^{s}_{w(e_i)}(z_i)$ and $\beta \in \varphi^{s}_{w(e_j)}(z_j)$ for $1 \leq i < j \leq p$, denote by u, v the two vertices, and $e' = z_i u$ and $e'' = z_j v$ the two edges such that $\varphi(e') = \alpha$ and $\varphi(e'') = \beta$, then we have $u \neq v$.*

Furthermore, if F^{ce} is maximal, i.e., there is no C-e-fan containing F^{ce} as a proper subsequence, then the following equation holds.

$$|C_{\varphi}(x)| + |C_{\varphi}(y)| = \sum_{z \in V(F^{ce}) \setminus \{x,y\}} (\mu_{F^{ce}}(x, z) + \mu_{F^{ce}}(y, z) - 2|C_{\varphi}(z)|),$$

where $\mu_{F^{ce}}(x, z)$ and $\mu_{F^{ce}}(y, z)$ taking value 0 or 1 are the number of edges between x and z and between y and z in F^{ce}, respectively.

3 Notation and Lemmas

Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. For a color $\alpha \in [\Delta]$, let $E_{\varphi,\alpha}(G)$ denote the set of edges colored with α. Let $\alpha, \beta \in [\Delta]$ be two distinct colors and H be the spanning subgraph induced by $E_{\varphi,\alpha}(G)$ and $E_{\varphi,\beta}(G)$. Clearly, every component of H is either a path or an even cycle which are referred as (α, β)-*chains* of G. If we interchange the colors α and β on (α, β)-chain C, then we obtain a new edge Δ-coloring of G, wrote by φ/C, which is also in $\mathcal{C}^{\Delta}(G-e)$. This operation is called a *Kempe change*. Furthermore, we say that a chain C has *endvertices* u and v if C is a path joining vertices u and v. For a vertex v of G, we denote by $P_v(\alpha, \beta, \varphi)$ the unique (α, β)-chain containing the vertex v. For two vertices u, $v \in V(G)$, the two chains $P_u(\alpha, \beta, \varphi)$ and $P_v(\alpha, \beta, \varphi)$ are either identical or disjoint.

Lemma 3.1. [?] *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. And let $F = (x, e, y_0, e_1, y_1, \dots, e_p, y_p)$ be a multi-fan at x, where $y_0 = y$. Then the following statements hold.*

(*a*) *$V(F)$ is elementary.*

(*b*) *If $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y_i)$ for $0 \leq i \leq p$, then $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.*

The following lemma is a simple corollary of Lemma 3.1.

Lemma 3.2. [?] *Let G be a Δ-critical graph. Then for any edge $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$, we have $d(x) + d(y) \geq \Delta + 2$.*

Lemma 3.3. [?] *Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. And let $F^c = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a C-fan at x, where $y_0 = y$. Then the following statements hold.*

(a) $V(F^c)$ is φ^c-elementary, i.e., $C_{\varphi,x}(x) \cap C_{\varphi,x}(y_i) = \emptyset$ for $i = 0, 1, \ldots, p$, and $C_{\varphi,x}(y_i) \cap C_{\varphi,x}(y_j) = \emptyset$ for $0 \leq i < j \leq p$.

(b) If $\alpha \in C_{\varphi,x}(x)$ and $\beta \in C_{\varphi,x}(y_i)$ for $0 \leq i \leq p$, then $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.

In a Δ-critical graph G with $e = xy \in E(G)$, a vertex u is called a *small vertex* with respect to x (with respect to y, respectively) if $d(u) \leq \frac{\Delta - d(x)}{2}$ ($d(u) \leq \frac{\Delta - d(y)}{2}$, respectively). We list the following simple facts [**?**].

Lemma 3.4. *In a Δ-critical graph G with $e = xy \in E(G)$, for any small vertices u, v with respect to x (with respect to y, respectively), we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \geq 1$ ($|\overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \geq 1$, respectively). In particular, provided $d(x) \leq d(y)$, no matter u and v are small vertices with respect to x or y, then we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \geq 1$. Furthermore, if $d(x) \leq \Delta(G) - 1$ and u is a small vertex with respect to x ($d(y) \leq \Delta(G) - 1$ and u is a small vertex with respect to y, respectively), then we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u)| \geq 2$ ($|\overline{\varphi}(y) \cap \overline{\varphi}(u)| \geq 2$, respectively).*

4 Proof of Theorem 1.3

In a simple e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, a *linear e-sequence* is a subsequence $(x, e, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ with $1 \leq l_1 < l_2 < \cdots < l_s \leq p$ such that $\varphi(e_{l_1}) \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ and $\varphi(e_{l_i}) \in \overline{\varphi}(z_{l_{i-1}})$ for $2 \leq i \leq s$. Specifically, a linear e-sequence is a *x-generated e-sequence* if $\varphi(e_{l_1}) \in \overline{\varphi}(x)$, or a *$y$-generated e-sequence* if $\varphi(e_{l_1}) \in \overline{\varphi}(y)$.

Proof. In the e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $z_i = z_j$ with $1 \leq i < j \leq p$, we delete the edge e_j and the vertex z_j from F^e. By the definition of e-fan, one can easily check that the remaining sequence is still an e-fan. Repeat the above operation. Finally, we get a simple e-fan F'^e with respect to the e-fan F^e. Obviously, $V(F^e) = V(F'^e)$. Hence, we may assume that the original e-fan F^e is a simple e-fan. We show the following two claims.

Claim 1. The vertex set of any linear e-sequence is elementary.

Proof. Suppose that Claim 1 is false. Without loss of generality, we choose φ such that there exists a y-generated e-sequence $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_s}, z_{l_s})$ with $e_{l_0} = e$, whose vertex set is not elementary with s as small as possible. Note that $e_{l_1} = xz_{l_1}$. Let $\varphi(e_{l_1}) = \beta_{l_1} \in \overline{\varphi}(y)$ and $\varphi(e_{l_i}) = \beta_{l_i} \in \overline{\varphi}(z_{l_{i-1}})$ for $2 \le i \le s$.

If $s \le 1$, then S_y is a Vizing fan at x, which has elementary vertex set by Lemma 3.1. We assume $s \ge 2$. By the minimality of s, $V(S_y)\backslash\{z_{l_s}\}$ is elementary. Together with the definition of y-generated e-sequence, we have that for any color $\gamma_1 \in \overline{\varphi}(x)$, no edge in $E(S_y)$ is colored with γ_1; for any color γ_2, if $\gamma_2 \in \overline{\varphi}(y)$ or $\gamma_2 \in \overline{\varphi}(z_{l_i})$ for $1 \le i \le s-1$, then only the edge e_{l_1} or $e_{l_{i+1}}$ in $E(S_y)$ may be colored with γ_2. We will use above facts about S_y without explicit mention. The following observation will also be used very often.

I. For any two colors $\gamma_1 \in \overline{\varphi}(x)$ and $\gamma_2 \in \overline{\varphi}(z_{l_i})$ with $1 \le i \le s-1$, we have $\gamma_1 \neq \gamma_2$ and $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$.

Proof. Recall that $V(S_y)\backslash\{z_{l_s}\}$ is elementary. We easily have $\gamma_1 \neq \gamma_2$. Suppose $P_x(\gamma_1, \gamma_2, \varphi) \neq P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$. For the path $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$, one endvertex is z_{l_i} and the other endvertex is some vertex $z' \neq x$. Note that $z' \notin \{y, z_{l_1}, \dots, z_{l_{i-1}}\}$ and none of $e_{l_1}, \dots, e_{l_i}$ is colored with γ_1 or γ_2. Hence, the coloring $\varphi' = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ satisfies $\varphi'(e_{l_j}) = \varphi(e_{l_j})$ for each $j \in [i]$, $\overline{\varphi}'(x) = \overline{\varphi}(x)$, $\overline{\varphi}'(y) = \overline{\varphi}(y)$, $\overline{\varphi}'(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [i-1]$ and $\overline{\varphi}'(z_{l_i}) = (\overline{\varphi}(z_{l_i})\backslash\{\gamma_2\}) \cup \{\gamma_1\}$. Consequently, the coloring φ' results in a new y-generated e-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}'(z_{l_i}) \cap \overline{\varphi}'(x)$, contradicting the minimality of s. This completes the proof of the observation I. □

Subclaim 1.1. We may assume that $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) \neq \emptyset$.

Proof. Since $V(S_y)$ is not elementary, and by the minimality of s, there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{x, y, z_{l_1}, \dots, z_{l_{s-1}}\})$. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$, then we are done. Otherwise, we have $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) = \emptyset$ and $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{y, z_{l_1}, \dots, z_{l_{s-1}}\})$ i.e., $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(y)$ or $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{z_{l_1}, \dots, z_{l_{s-1}}\})$. By the definition of S_y, we have $\eta \neq \beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$. Let $\alpha \in \overline{\varphi}(x)$. Since $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) = \emptyset$, we have $\alpha \neq \eta$ and $\alpha \in \varphi(z_{l_s})$. Note that if $\eta \in \overline{\varphi}(y)$, then $P_x(\alpha, \eta, \varphi) = P_y(\alpha, \eta, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Also if $\eta \in \overline{\varphi}(z_{l_i})$, then $P_x(\alpha, \eta, \varphi) = P_{z_{l_i}}(\alpha, \eta, \varphi)$ for $1 \le i \le s-1$ by the observation I. Therefore, $P_x(\alpha, \eta, \varphi)$ and $P_{z_{l_s}}(\alpha, \eta, \varphi)$ are disjoint. For the path $P = P_{z_{l_s}}(\alpha, \eta, \varphi)$, one endvertex is z_{l_s} and the other endvertex $z' \notin V(S_y)$, and we have $E_{\varphi,\alpha}(P) \cap E(S_y) = \emptyset$. Note that if $\eta = \beta_{l_1} \in \overline{\varphi}(y)$, then e_{l_1} is on $P_x(\alpha, \eta, \varphi)$. To further discuss $E_{\varphi,\eta}(P) \cap E(S_y)$, we consider the following two cases.

If $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is on P for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \le i \le s-2$, then we have $E_{\varphi,\eta}(P) \cap E(S_y) = \{e_{l_{i+1}}\}$. Hence, the coloring $\varphi_1 = \varphi/P$ satisfies $\varphi_1(e_{l_j}) = \varphi(e_{l_j})$ for $j \neq i$, $\varphi_1(e_{l_{i+1}}) = \alpha$, $\overline{\varphi}_1(x) = \overline{\varphi}(x)$, $\overline{\varphi}_1(y) = \overline{\varphi}(y)$, $\overline{\varphi}_1(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [s-1]$

and $\overline{\varphi}_1(z_{l_s}) = (\overline{\varphi}(z_{l_s})\backslash\{\eta\}) \cup \{\alpha\}$. Consequently, the coloring φ_1 results in a smaller x-generated e-sequence $(x, e_{l_0}, y, e_{l_{i+1}}, z_{l_{i+1}}, \dots, e_{l_s}, z_{l_s})$ with $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s.

If $\eta \in \overline{\varphi}(y)$, or $\eta \neq \beta_{l_{i+1}}$ for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \leq i \leq s-1$, or $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is not on P for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \leq i \leq s-2$, then we have $E_{\varphi,\eta}(P) \cap E(S_y) = \emptyset$. Hence, the coloring $\varphi_1 = \varphi/P$ satisfies $\varphi_1(e_{l_j}) = \varphi(e_{l_j})$ for each $j \in [s]$, $\overline{\varphi}_1(x) = \overline{\varphi}(x)$, $\overline{\varphi}_1(y) = \overline{\varphi}(y)$, $\overline{\varphi}_1(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [s-1]$ and $\overline{\varphi}_1(z_{l_s}) = (\overline{\varphi}(z_{l_s})\backslash\{\eta\}) \cup \{\alpha\}$. Consequently, S_y is still a y-generated e-sequence with $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. This completes the proof of Subclaim 1.1. □

By the subclaim above, we assume that there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$. To reach contradictions, we consider the following two cases.

Case 1. $e_{l_s} = xz_{l_s}$.

Note that $\varphi(e_{l_s}) = \beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$ and none of $e_{l_1}, \dots, e_{l_{s-1}}$ is colored with β_{l_s} or η. Recolor e_{l_s} with η to obtain a new coloring φ_1. Thus $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_{s-1}}, z_{l_{s-1}})$ is a new y-generated e-sequence under φ_1 such that $\beta_{l_s} \in \overline{\varphi}_1(z_{l_{s-1}}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s.

Case 2. $e_{l_s} = yz_{l_s}$.

By the observation I, we have $P_x(\eta, \beta_{l_s}, \varphi) = P_{z_{l_{s-1}}}(\eta, \beta_{l_s}, \varphi)$. For the path $P = P_{z_{l_s}}(\eta, \beta_{l_s}, \varphi)$, one endvertex is z_{l_s} and the other endvertex $z' \notin V(S_y)$, and we have $E(P) \cap E(S_y) = \{e_{l_s}\}$. Let $\varphi_1 = \varphi/P$. Hence $(x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_{s-1}}, z_{l_{s-1}})$ is still a y-generated e-sequence under φ_1 whose vertex set is still elementary, and $(y, e_{l_0}, x, e_{l_s}, z_{l_s})$ is a Vizing fan at y under φ_1 since $\varphi_1(e_{l_s}) = \eta \in \overline{\varphi}_1(x)$. Since $min\{d(x), d(y)\} \leq \Delta - 1$, there exists a missing color $\delta \in \overline{\varphi}_1(x) \cup \overline{\varphi}_1(y)$ such that $\delta \neq \eta, \beta_{l_1}$. Suppose $\delta \in \overline{\varphi}_1(x)$. We have $P_x(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ by Lemma 3.1, since otherwise, the coloring $\varphi' = \varphi_1/P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ results in $\delta \in \overline{\varphi}'(z_{l_s}) \cap \overline{\varphi}'(x)$, which is a contradiction. But we have $P_x(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_{s-1}}}(\delta, \beta_{l_s}, \varphi_1)$ by the observation I, giving a contradiction. Similarly, if $\delta \in \overline{\varphi}_1(y)$, then $P_y(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ by Lemma 3.1. But $P_y(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_{s-1}}}(\delta, \beta_{l_s}, \varphi_1)$, also giving a contradiction. This completes the proof of Claim 1. □

Claim 2. The union of vertex sets of any two linear e-sequences is elementary.

Proof. Suppose that Claim 2 is false. Without loss of generality, we choose φ such that there exist two linear e-sequences $S_1 = (x, e, y, e_{l_1}, z_{l_1}, \dots, e_{l_s}, z_{l_s})$ and $S_2 = (x, e, y, e_{l'_1}, z_{l'_1}, \dots, e_{l'_s}, z_{l'_t})$ whose vertex sets have common missing color with $s + t$ as small as possible, where $s, t \geq 1$. Note that $V(S_1)$ and $V(S_2)$ are elementary by Claim 1. By the minimality of $s + t$, we have $z_{l_s} \neq z_{l'_t}$ and there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l'_t})$. Since $min\{d(x), d(y)\} \leq \Delta - 1$, there exists a missing color $\delta \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ such that δ is different from the colors $\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ which are also in $\overline{\varphi}(x) \cup \overline{\varphi}(y)$. ($\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ could be the same color.) Assume $\delta \in \overline{\varphi}(z_0)$, where $z_0 \in \{x, y\}$. Then $P_{z_0}(\delta, \eta, \varphi) = P_{z_{l_s}}(\delta, \eta, \varphi)$, since otherwise, for the coloring $\varphi' = \varphi / P_{z_{l_s}}(\delta, \eta, \varphi)$, we have S_1 is still a linear e-sequence under φ', but $\delta \in \overline{\varphi}'(z_0) \cap \overline{\varphi}'(z_{l_s})$, giving a contradiction to Claim 1. Similarly, we have $P_{z_0}(\delta, \eta, \varphi) = P_{z_{l'_t}}(\delta, \eta, \varphi)$. Hence z_0, z_{l_s} and $z_{l'_t}$ are endvertices of one (δ, η)-chain, which is a contradiction. This completes the proof of Claim 2. □

Now we are ready to show that $V(F^e)$ is elementary. Suppose not. Note that $\{x, y\}$ is elementary and each linear e-sequence in F^e contains vertices x and y. There exist one color η and two distinct vertices z_i and z_j in $V(F^e)$ such that $\eta \in \overline{\varphi}(z_i) \cap \overline{\varphi}(z_j)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$. By the definition of simple e-fan, there exist two linear e-sequences (may not be disjoint) with z_i and z_j respectively as the last vertex, which is a contradiction to Claim 1 for $i = 0$ or a contradiction to Claim 2 for $1 \leq i \leq p - 1$. This proves that $V(F^e)$ is elementary.

Now we show the "furthermore" part. We assume that F^e is maximal. Let the edge set $\Gamma = \{e_1, \ldots, e_p\}$ and the color set $\Gamma' = \bigcup_{z \in V(F^e)} \overline{\varphi}(z)$. Note that $\overline{\varphi}(x)$, $\overline{\varphi}(y)$ and $\overline{\varphi}(z_i)$ for each $i \in [p]$ are disjoint since $V(F^e)$ is elementary. Let $\Gamma^* = \{\varphi(e_1), \ldots, \varphi(e_p)\}$ be a multiset. We have

$$p = |\Gamma| = \sum_{z \in V(F^e) \setminus \{x,y\}} (\mu_{F^e}(x, z) + \mu_{F^e}(y, z)) = |\Gamma^*|. \tag{1}$$

Now we calculate $|\Gamma^*|$ in another way. By the definition of e-fan, $\varphi(e_i) \in \Gamma'$ for each $i \in [p]$. By the maximality of F^e, for any color $\alpha \in \Gamma'$, α appears exactly once in Γ^* if $\alpha \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$. Otherwise, α appears exactly twice in Γ^*. Thus we have

$$|\Gamma^*| = |\overline{\varphi}(x)| + |\overline{\varphi}(y)| + \sum_{z \in V(F^e) \setminus \{x,y\}} 2|\overline{\varphi}(z)|. \tag{2}$$

Combining equations (1) and (2), we prove that

$$d(x) + d(y) - 2\Delta + \sum_{z \in V(F^e) \setminus \{x,y\}} (2d(z) + \mu_{F^e}(x, z) + \mu_{F^e}(y, z) - 2\Delta) = 2,$$

since $\overline{\varphi}(x) = \Delta - d(x) + 1$, $\overline{\varphi}(y) = \Delta - d(y) + 1$ and $\overline{\varphi}(z) = \Delta - d(z)$. □

5 Proof of Theorem 2.3

Note that when $d(x) \neq d(y)$ the values of $|C_{\varphi, w(e_i)}(z_i)|$ and $|C_{\varphi, w(e_j)}(z_j)|$ may not be equal for repeated vertices $z_i = z_j$ with $i \neq j$ in C-e-fan F^{ce}. We define *simple C-e-fan* if we further require that vertices $x, y, z_1, \dots, z_p$ are distinct except the repeated vertices $z_i = z_j$ with $1 \leq i < j \leq p$ such that $C_{\varphi,w(e_i)}(z_i) \subset C_{\varphi,w(e_j)}(z_j)$ in the definition of C-e-fan. In a simple C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \dots, e_p, z_p)$, a *linear ce-sequence* is a subsequence $(x, e, y, e_{l_1}, z_{l_1}, \dots, e_{l_s}, z_{l_s})$ with $1 \leq l_1 < l_2 < \dots < l_s \leq p$ such that $\varphi(e_{l_1}) \in C_{\varphi,y}(x) \cup C_{\varphi,x}(y)$ and $\varphi(e_{l_i}) \in C_{\varphi,w(e_{l_{i-1}})}(z_{l_{i-1}})$ for $2 \leq i \leq s$. Specifically, a linear ce-sequence is a *x-generated ce-sequence* if $\varphi(e_{l_1}) \in C_{\varphi,y}(x)$, or a *$y$-generated ce-sequence* if $\varphi(e_{l_1}) \in C_{\varphi,x}(y)$.

Proof. In the C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \dots, e_p, z_p)$, if $z_i = z_j$ with $1 \leq i < j \leq p$ and $C_{\varphi,w(e_i)}(z_i) \supseteq C_{\varphi,w(e_j)}(z_j)$, we delete the edge e_j and the vertex z_j from F^{ce}. By the definition of C-e-fan, one can easily check that the remaining sequence is still a C-e-fan. Repeat the above operation. Finally, we get a simple C-e-fan F'^{ce} with respect to the C-e-fan F^{ce}. Obviously, $V(F^{ce}) = V(F'^{ce})$ and the $C_\varphi(u)$ in F^{ce} is the same as the $C_\varphi(u)$ in F'^{ce} for each vertex u. Hence, we may assume that the original C-e-fan F^{ce} is a simple C-e-fan. We show the following two claims.

Claim 1. The vertex set of any linear ce-sequence is φ^{ce}-elementary.

Proof. Suppose that Claim 1 is false. Without loss of generality, we choose φ such that there exists a y-generated ce-sequence $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_s}, z_{l_s})$ with $e_{l_0} = e$, whose vertex set is not φ^{ce}-elementary with s as small as possible. Note that $e_{l_1} = xz_{l_1}$. Let $\varphi(e_{l_1}) = \beta_{l_1} \in C_{\varphi,x}(y)$ and $\varphi(e_{l_i}) = \beta_{l_i} \in C_{\varphi,w(e_{l_{i-1}})}(z_{l_{i-1}})$ for $2 \leq i \leq s$. We consider the following two cases of s.

First we consider the case $s \leq 1$. It is easy to see that S_y is a C-fan at x. By the statement (a) of Lemma 3.3, we have $C_{\varphi,x}(x) \cap C_{\varphi,x}(y) = \emptyset$, $C_{\varphi,x}(x) \cap C_{\varphi,x}(z_{l_1}) = \emptyset$ and $C_{\varphi,x}(y) \cap C_{\varphi,x}(z_{l_1}) = \emptyset$. Recall that $C_{\varphi,x}(x) = \overline{\varphi}(x)$. Since we suppose that Claim 1 is false, there are four subcases left to consider.

If there exists $\eta \in \varphi_y^s(x) \cap \overline{\varphi}(y)$, then it contradicts Lemma 3.3 since C-fan (y, e_{l_0}, x) at y. If there exists $\eta \in \varphi_y^s(x) \cap \varphi_x^s(y)$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = yv$ such that $v \neq x$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(x)}{2}$. Obviously, $u \neq v$. Recall that $max\{d(x), d(y)\} \leq \Delta - 1$. It follows from Lemma 3.4 that there are two colors $\delta_1 \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$ and $\delta_2 \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta_2 \neq \beta_{l_1}$. We have $\delta_1 \neq \delta_2$ and $P_x(\delta_1, \delta_2, \varphi) = P_y(\delta_1, \delta_2, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Do Kempe changes on $P_u(\delta_1, \delta_2, \varphi)$ and $P_v(\delta_1, \delta_2, \varphi)$ to get a new coloring φ_1 such that $\delta_1 \in \overline{\varphi}_1(x) \cap \overline{\varphi}_1(u)$ and $\delta_2 \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(v)$. Recolor the edge e' with δ_1 and the edge e'' with δ_2 to get a new coloring φ_2 such that $\eta \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(y)$. Now by coloring the edge e with η, we color the entire graph G with Δ colors, which contradicts the fact that $\chi'(G) = \Delta + 1$.

If there exists $\eta \in \varphi_y^s(x) \cap \overline{\varphi}(z_{l_1})$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(y)}{2}$. Since $max\{d(x), d(y)\} \leq \Delta - 1$, it follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta \neq \beta_{l_1}$. We have $x \in P_y(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 3.3 since C-fan (y, e_{l_0}, x) at y. Recall that $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1})$ is a C-fan at x. The coloring $\varphi_1 = \varphi/P_{z_{l_1}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_1}) \cap \overline{\varphi}_1(y)$, which contradicts Lemma 3.3 because S_y is still a C-fan at x under φ_1.

If there exists $\eta \in \varphi^s_y(x) \cap \varphi^s_x(z_{l_1})$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = z_{l_1}v$ such that $v \neq x$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(x)}{2}$. Obviously, $u \neq v$, and we have $v \neq y$ by Lemma 3.2. By Lemma 3.4, there are two colors $\delta_1 \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$ and $\delta_2 \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta_2 \neq \beta_{l_1}$. We have $P_x(\delta_1, \delta_2, \varphi) = P_y(\delta_1, \delta_2, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Do Kempe changes on $P_u(\delta_1, \delta_2, \varphi)$ and $P_v(\delta_1, \delta_2, \varphi)$ to get a new coloring φ_1 such that $\delta_1 \in \overline{\varphi}_1(x) \cap \overline{\varphi}_1(u)$ and $\delta_2 \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(v)$. Note that $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1})$ is still a C-fan at x under φ_1. Recolor the edge e' with δ_1 to get a new coloring φ_2. Thus $\eta \in \overline{\varphi}_2(x) \cap C_{\varphi_2, x}(z_{l_1})$, which contradicts Lemma 3.3 because S_y is still a C-fan at x under φ_2. This completes the proof of Claim 1 for $s \leq 1$.

Now we consider the case $s \geq 2$. By the minimality of s, $V(S_y \backslash \{e_{l_s}, z_{l_s}\})$ is φ^{ce}-elementary. Together with the definition of y-generated ce-sequence, we have that for any color $\gamma_1 \in C_{\varphi,y}(x)$, no edge in $E(S_y)$ is colored with γ_1; for any color γ_2, if $\gamma_2 \in C_{\varphi,x}(y)$ or $\gamma_2 \in C_{\varphi, w(e_{l_i})}(z_{l_i})$ with $1 \leq i \leq s-1$, where z_{l_i} is not a repeated vertex, then only the edge e_{l_1} or $e_{l_{i+1}}$ in $E(S_y)$ may be colored with γ_2; for any color $\gamma_3 \in C_\varphi(z)$, where z is a repeated vertex with $z = z_{l_i} = z_{l_j}$ and $1 \leq i < j \leq s-1$, only the edge $e_{l_{i+1}}$ or $e_{l_{j+1}}$ in $E(S_y)$ may be colored with γ_3. We will use above facts about S_y without explicit mention. The following observation will also be used very often.

II. For any color γ_1 with $\gamma_1 \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ and $\gamma_1 \neq \beta_{l_1}$, if color $\gamma_2 \in \overline{\varphi}(z_{l_i})$ with $1 \leq i \leq s-1$, then we have $\gamma_1 \neq \gamma_2$ and $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ or $P_y(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$; if $\gamma_2 \in \varphi^s_{w(e_{l_i})}(z_{l_i})$ with $1 \leq i \leq s-1$, denote by u the vertex and $e' = z_{l_i}u$ the edge such that $\varphi(e') = \gamma_2$, and further provide $\gamma_1 \in \overline{\varphi}(u)$, then we have $z_{l_i} \in P_x(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$ or $z_{l_i} \in P_y(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$.

Proof. We first assume $\gamma_1 \in \overline{\varphi}(x)$. Recall that $V(S_y \backslash \{e_{l_s}, z_{l_s}\})$ is φ^{ce}-elementary. We easily have $\gamma_1 \neq \gamma_2$ and $\gamma_2 \in \varphi(x)$. Suppose $P_x(\gamma_1, \gamma_2, \varphi) \neq P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_x(\gamma_1, \gamma_2, \varphi) \neq P_u(\gamma_1, \gamma_2, \varphi)$, respectively). For the path $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively), one endvertex is z_{l_i} (u, respectively) and the other endvertex is some vertex $z' \neq x$. Note that $z' \notin \{y, z_{l_1}, \ldots, z_{l_{i-1}}\}$ and none of $e_{l_1}, \ldots, e_{l_i}$ is colored with γ_1. Since z_{l_i} may be a repeated vertex in S_y, we consider the following two cases. If z_{l_i} is not a repeated vertex or z_{l_i} is a repeated vertex with $z_{l_i} = z_{l_k}$ and $1 \leq i < k \leq s-1$, then none of $e_{l_1}, \ldots, e_{l_i}$ is colored with γ_2. Hence, the coloring $\varphi_1 = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi_1 = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively) results in a new y-generated ce-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}_1(z_{l_i}) \cap \overline{\varphi}_1(x)$ ($\gamma_1 \in C_{\varphi_1, w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}_1(x)$, respectively), contradicting the minimality of s.

If z_{l_i} is a repeated vertex with $z_{l_k} = z_{l_i}$ and $1 \le k < i \le s-1$, then only the edge $e_{l_{k+1}}$ of $e_{l_1}, \dots, e_{l_i}$ may be colored with γ_2. We claim that $e_{l_{k+1}}$ is not on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively). If $\varphi(e_{l_{k+1}}) \ne \gamma_2$, then we are done. If $\varphi(e_{l_{k+1}}) = \gamma_2$ and $e_{l_{k+1}} = xz_{l_{k+1}}$, then $e_{l_{k+1}}$ is on $P_x(\gamma_1, \gamma_2, \varphi)$, and we are also done. If $\varphi(e_{l_{k+1}}) = \gamma_2$, $e_{l_{k+1}} = yz_{l_{k+1}}$ and $e_{l_{k+1}}$ is on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively), then the coloring $\varphi' = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi' = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively) results in a smaller x-generated ce-sequence $(x, e_{l_0}, y, e_{l_{k+1}}, z_{l_{k+1}}, \dots, e_{l_i}, z_{l_i})$ since $\varphi'(e_{l_{k+1}}) = \gamma_1 \in \overline{\varphi}'(x)$ such that $\gamma_1 \in \overline{\varphi}'(z_{l_i}) \cap \overline{\varphi}'(x)$ ($\gamma_1 \in C_{\varphi', w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}'(x)$, respectively), contradicting the minimality of s. Now we have that $e_{l_{k+1}}$ is not on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively). Let the coloring $\varphi_1 = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi_1 = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively), which results in a new y-generated ce-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \dots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}_1(z_{l_i}) \cap \overline{\varphi}_1(x)$ ($\gamma_1 \in C_{\varphi_1, w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}_1(x)$, respectively), also contradicting the minimality of s. This completes the proof of $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_x(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$, respectively). Similarly, we have $P_y(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_y(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$, respectively) for $\gamma_1 \in \overline{\varphi}(y)$ and $\gamma_1 \ne \beta_{l_1}$. □

By the minimality of s, we have that either z_{l_s} is not a repeated vertex or z_{l_s} is a repeated vertex with $z_{l_k} = z_{l_s}$ and $C_{\varphi, w(e_{l_k})}(z_{l_k}) \subset C_{\varphi, w(e_{l_s})}(z_{l_s})$, where $1 \le k < s$. By the minimality of s again, there exists a color $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap (C_{\varphi, y}(x) \cup C_{\varphi, x}(y) \cup C_{\varphi, w(e_{l_i})}(z_{l_i}))$ with $1 \le i \le s-1$. And if z_{l_s} is a repeated vertex with $z_{l_k} = z_{l_s}$ and $1 \le k < s$, then we have $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \backslash C_{\varphi, w(e_{l_k})}(z_{l_k}) = \varphi^s_{w(e_{l_s})}(z_{l_s}) \backslash \varphi^s_{w(e_{l_k})}(z_{l_k})$. Let $\alpha \in \overline{\varphi}(x)$.

Subclaim 1.1. We may assume that $C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x) \ne \emptyset$.

Proof. In order to prove the above subclaim, we consider the following three cases.

Case 1. $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap C_{\varphi, y}(x)$.

If $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$, then we are done. Otherwise, first suppose $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_y(x)$, then there is an edge $e' = xu$ such that $u \ne y$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(y)}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta \ne \beta_{l_1}$. We have $x \in P_y(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 3.3 since C-fan (y, e_{l_0}, x) at y. The coloring $\varphi_1 = \varphi/P_{z_{l_s}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_s})$ and S_y is still a y-generated ce-sequence under φ_1. We have $P_x(\alpha, \delta, \varphi_1) = P_y(\alpha, \delta, \varphi_1)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y) under φ_1. Then the coloring $\varphi_2 = \varphi_1/P_{z_{l_s}}(\alpha, \delta, \varphi_1)$ results in $\alpha \in \overline{\varphi}_2(z_{l_s}) \cap \overline{\varphi}_2(x)$, which is as desired because S_y is still a y-generated ce-sequence under φ_2 and $C_{\varphi_2, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_2(x) \ne \emptyset$.

Now suppose $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_y(x)$. Thus there is an edge $e' = xu$ such that $u \ne y$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = z_{l_s}v$ such that $v \ne w(e_{l_s})$, $\varphi(e'') = \eta$ and $d(v) \le \frac{\Delta - d(w(e_{l_s}))}{2}$. Obviously, $u \ne v$. We consider the following two subcases. If $d(x) \le d(y)$, then by Lemma 3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. Recolor the edge e' with δ to get a new coloring φ_1 such that $\eta \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$. Then we are done because S_y is still a y-generated ce-sequence under φ_1 and $C_{\varphi_1, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x) \neq \emptyset$. If $d(x) > d(y)$, then by Lemma 3.4, there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Do Kempe changes on $P_u(\alpha, \delta, \varphi)$ and $P_v(\alpha, \delta, \varphi)$ to get a new coloring φ_2 such that $\alpha \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(u) \cap \overline{\varphi}_2(v)$. Since S_y is still a y-generated ce-sequence under φ_2, we are in the previous subcase in this paragraph with α in place of δ.

Case 2. $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap C_{\varphi, x}(y)$.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(y)$, then we have $P_x(\alpha, \eta, \varphi) = P_y(\alpha, \eta, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\alpha, \eta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi / P_{z_{l_s}}(\alpha, \eta, \varphi)$ results in $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired because S_y is still a y-generated ce-sequence under φ_1 and $C_{\varphi_1, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x) \neq \emptyset$.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_x(y)$, then there is an edge $e' = yu$ such that $u \neq x$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(x)}{2}$. By Lemma 3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. We have $y \in P_x(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 3.3 since C-fan $(x, e_{l_0}, y, e_{l_1}, z_{l_1})$ at x. Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi / P_{z_{l_s}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(y)$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ with $\delta \neq \eta$. We consider the following two subcases. If $w(e_{l_s}) = x$, then we have $P_x(\eta, \delta, \varphi) = P_y(\eta, \delta, \varphi)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi / P_u(\eta, \delta, \varphi)$ results in $\delta \in (\varphi_1)^s_x(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. If $w(e_{l_s}) = y$, then we have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 3.1. Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Then the coloring $\varphi_2 = \varphi / P_u(\alpha, \delta, \varphi)$ results in $\alpha \in \overline{\varphi}_2(u)$. We have $P_x(\alpha, \eta, \varphi_2) = P_y(\alpha, \eta, \varphi_2)$ by Lemma 3.1 since Vizing fan (x, e_{l_0}, y) under φ_2. Then the coloring $\varphi_3 = \varphi_2 / P_u(\alpha, \eta, \varphi_2)$ results in $\alpha \in (\varphi_3)^s_y(z_{l_s}) \cap \overline{\varphi}_3(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_x(y)$, then there is an edge $e' = yu$ such that $u \neq x$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(x)}{2}$, and there is an edge $e'' = z_{l_s}v$ such that $v \neq w(e_{l_s})$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. Obviously, $u \neq v$. We consider the following two subcases. If $d(x) \leq d(y)$, then by Lemma 3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $y \in P_x(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 3.3 since C-fan $(x, e_{l_0}, y, e_{l_1}, z_{l_1})$ at x. Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi / P_v(\eta, \delta, \varphi)$ results in $\delta \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. If $d(x) > d(y)$, then by Lemma 3.4, there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 3.1. Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Do Kempe changes on $P_u(\alpha, \delta, \varphi)$ and $P_v(\alpha, \delta, \varphi)$ to get a new coloring φ_2 such that $\alpha \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(u) \cap \overline{\varphi}_2(v)$. Thus we are in the previous subcase in this paragraph with α in place of δ.

Case 3. $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap C_{\varphi,w(e_{l_i})}(z_{l_i})$ for $1 \le i \le s-1$.

By the minimality of s, we have $z_{l_s} \neq z_{l_i}$. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$, then $P_x(\alpha,\eta,\varphi) = P_{z_{l_i}}(\alpha,\eta,\varphi)$ by the observation II. For the path $P = P_{z_{l_s}}(\alpha,\eta,\varphi)$, one endvertex is z_{l_s}, the other endvertex is $z' \notin V(S_y)$ and $E_{\varphi,\alpha}(P) \cap E(S_y) = \emptyset$. In order to do Kempe change on P, we should discuss the following $E_{\varphi,\eta}(P) \cap E(S_y)$. Let $z_{l_i} = z_{l_j}$ with $1 \le i \neq j \le s-1$ if z_{l_i} is a repeated vertex in S_y. Note that only one of $e_{l_{i+1}}$, $e_{l_{j+1}}$ may be colored with η. We consider the following two subcases. If $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is on P (or $\eta = \beta_{l_{j+1}}$ and $e_{l_{j+1}}$ is on P by symmetry), then $E_{\varphi,\eta}(P) \cap E(S_y) = \{e_{l_{i+1}}\}$ and the coloring $\varphi_1 = \varphi/P$ results in a smaller x-generated ce-sequence $(x, e_{l_0}, y, e_{l_{i+1}}, z_{l_{i+1}}, \ldots, e_{l_s}, z_{l_s})$ since $\varphi_1(e_{l_{i+1}}) = \alpha \in \overline{\varphi}_1(x)$ such that $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s. If $\eta \neq \beta_{l_{i+1}}, \beta_{l_{j+1}}$, or $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is not on P, then $E_{\varphi,\eta}(P) \cap E(S_y) = \emptyset$ and the coloring $\varphi_1 = \varphi/P$ results in $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired because S_y is still a y-generated ce-sequence under φ_1.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ and $\delta \neq \beta_{l_1}$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \neq \emptyset$. If $w(e_{l_s}) = x$, then we are done. Otherwise, consider the case $w(e_{l_s}) = y$. We have $P_x(\alpha,\delta,\varphi) = P_y(\alpha,\delta,\varphi)$ by Lemma 3.1. Then the coloring $\varphi' = \varphi/P_u(\alpha,\delta,\varphi)$ results in $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. By the observation II, we have $P_x(\gamma,\eta,\varphi) = P_{z_{l_i}}(\gamma,\eta,\varphi)$. By the similar proof of the first subcase of Case 3 (i.e., the case $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$) with $P_u(\gamma,\eta,\varphi)$ in place of P and γ in place of α, we can obtain the coloring $\varphi_1 = \varphi/P_u(\gamma,\eta,\varphi)$ such that $\gamma \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_{w(e_{l_i})}(z_{l_i})$, then there is an edge $e' = z_{l_i}u$ such that $u \neq w(e_{l_i})$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(w(e_{l_i}))}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_i})) \cap \overline{\varphi}(u)$ with $\delta \neq \beta_{l_1}$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \neq \emptyset$. If $w(e_{l_i}) = x$, then we are done. Otherwise, consider the case $w(e_{l_i}) = y$. We have $P_x(\alpha,\delta,\varphi) = P_y(\alpha,\delta,\varphi)$ by Lemma 3.1. Then the coloring $\varphi' = \varphi/P_u(\alpha,\delta,\varphi)$ results in $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. By the observation II, we have $P_x(\gamma,\eta,\varphi) = P_u(\gamma,\eta,\varphi)$. By the similar proof of the first subcase of Case 3 with γ in place of α, we can obtain the coloring $\varphi_1 = \varphi/P_{z_{l_s}}(\gamma,\eta,\varphi)$ such that $\gamma \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_{w(e_{l_i})}(z_{l_i})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l_i}v$ such that $v \neq w(e_{l_i})$, $\varphi(e'') = \eta$ and $d(v) \le \frac{\Delta - d(w(e_{l_i}))}{2}$. Obviously, $u \neq v$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v) \neq \emptyset$. If $d(x) \le d(y)$, then it follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$, and so we are done. If $d(x) > d(y)$, then it follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha,\delta,\varphi) = P_y(\alpha,\delta,\varphi)$ by Lemma 3.1. Do Kempe changes on $P_u(\alpha,\delta,\varphi)$ and $P_v(\alpha,\delta,\varphi)$, and get a new coloring φ' such that $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u) \cap \overline{\varphi}'(v)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. By the observation

II, we have $P_x(\gamma,\eta,\varphi) = P_u(\gamma,\eta,\varphi)$. By the similar proof of the first subcase of Case 3 with $P_u(\gamma,\eta,\varphi)$ in place of P and γ in place of α, we can obtain the coloring $\varphi_1 = \varphi/P_u(\gamma,\eta,\varphi)$ such that $\gamma \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

Combining the above Cases 1, 2 and 3, we complete the proof of Subclaim 1.1. $\square$

Thus we assume that there exists a color $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$. We consider the following two cases.

Case 1. $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$.
Suppose $w(e_{l_s}) = x$. Recolor the edge e_{l_s} with η to get a new coloring φ_1. Thus $\beta_{l_s} \in \overline{\varphi}_1(x) \cap C_{\varphi_1,w(e_{l_{s-1}})}(z_{l_{s-1}})$, which contradicts the minimality of s. So we assume $w(e_{l_s}) = y$. Since $d(y) \le \Delta - 1$, there exists a missing color γ with $\gamma \neq \beta_{l_1}$. We have $P_x(\eta,\gamma,\varphi) = P_y(\eta,\gamma,\varphi)$ by Lemma 3.1. Let $\varphi_2 = \varphi/P_{z_{l_s}}(\eta,\gamma,\varphi)$, and we have $\gamma \in \overline{\varphi}_2(y) \cap \overline{\varphi}_2(z_{l_s})$. Recolor the edge e_{l_s} with γ to get a new coloring φ_3. Thus $\beta_{l_s} \in \overline{\varphi}_3(y) \cap C_{\varphi_3,w(e_{l_{s-1}})}(z_{l_{s-1}})$, also contradicting the minimality of s.

Case 2. $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$.
Suppose $\beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$. Since $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s})$, there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ with $\delta \neq \eta, \beta_{l_1}$. By the observation II, we have $P_{w(e_{l_s})}(\delta,\beta_{l_s},\varphi) = P_{z_{l_{s-1}}}(\delta,\beta_{l_s},\varphi)$. Note that e_{l_s} is on $P_{w(e_{l_s})}(\delta,\beta_{l_s},\varphi)$. Let $\varphi_1 = \varphi/P_u(\delta,\beta_{l_s},\varphi)$. Hence S_y is still a y-generated sequence under φ_1 with $\beta_{l_s} \in \overline{\varphi}_1(u)$. We claim that we may assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. If $w(e_{l_s}) = x$, we are done. Otherwise, $w(e_{l_s}) = y$. We have $P_x(\eta,\delta,\varphi_1) = P_y(\eta,\delta,\varphi_1)$ by Lemma 3.1. Recall $\delta \neq \beta_{l_1}$. The coloring $\varphi' = \varphi_1/P_x(\eta,\delta,\varphi)$ results in $\eta \in \overline{\varphi}'(y)$, as desired. Now we assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. We have $P_{w(e_{l_s})}(\eta,\beta_{l_s},\varphi_1) = P_u(\eta,\beta_{l_s},\varphi_1) = w(e_{l_s})z_{l_s}u$. Then the coloring $\varphi_2 = \varphi_1/P_{w(e_{l_s})}(\eta,\beta_{l_s},\varphi_1)$ results in $\beta_{l_s} \in \overline{\varphi}_2(w(e_{l_s})) \cap \overline{\varphi}_2(z_{l_{s-1}})$, contradicting the minimality of s.

Now we suppose $\beta_{l_s} \in \varphi^s_{w(e_{l_{s-1}})}(z_{l_{s-1}})$. In this case, there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l_{s-1}}v$ such that $v \neq w(e_{l_{s-1}})$, $\varphi(e'') = \beta_{l_s}$ and $d(v) \le \frac{\Delta - d(w(e_{l_{s-1}}))}{2}$. By the condition **C3** in Section 2, we have $u \neq v$. It follows from Lemma 3.4 that there is a color $\delta \in (\overline{\varphi}(w(e_{l_s})) \cup \overline{\varphi}(w(e_{l_{s-1}}))) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We first claim that we may assume that $\delta \in \overline{\varphi}(w(e_{l_s}))$ and $\delta \neq \beta_{l_1}$. Suppose $\delta \in \overline{\varphi}(w(e_{l_s}))$ but $\delta = \beta_{l_1}$. Thus $w(e_{l_s}) = y$. Recall that $max\{d(x), d(y)\} \le \Delta - 1$. Hence there exist $\gamma_1 \in \overline{\varphi}(x)$ with $\gamma_1 \neq \eta$ and $\gamma_2 \in \overline{\varphi}(y)$ with $\gamma_2 \neq \delta = \beta_{l_1}$. By Lemma 3.1, we have $P_x(\gamma_1,\delta,\varphi) = P_y(\gamma_1,\delta,\varphi)$ and $P_x(\gamma_1,\gamma_2,\varphi) = P_y(\gamma_1,\gamma_2,\varphi)$. Do Kempe changes on $P_u(\gamma_1,\delta,\varphi)$ and $P_v(\gamma_1,\delta,\varphi)$ to get a new coloring φ'. And then do Kempe changes on $P_u(\gamma_1,\gamma_2,\varphi')$ and $P_v(\gamma_1,\gamma_2,\varphi')$ to get a new coloring φ''. Consequently, we have $\gamma_2 \in \overline{\varphi}''(u) \cap \overline{\varphi}''(v)$, as desired because γ_2 is the desired color instead of δ. Now suppose $\delta \notin \overline{\varphi}(w(e_{l_s}))$. Thus we have $w(e_{l_s}) \neq w(e_{l_{s-1}})$ and $\delta \in \overline{\varphi}(w(e_{l_{s-1}}))$. Since $max\{d(x), d(y)\} \le \Delta - 1$, there exists a missing color $\gamma \in \overline{\varphi}(w(e_{l_s}))$ such that $\gamma \neq \delta, \beta_{l_1}$. We have $P_x(\gamma, \delta, \varphi) = P_y(\gamma, \delta, \varphi)$ by Lemma 3.1. Do Kempe changes on $P_u(\gamma, \delta, \varphi)$ and $P_v(\gamma, \delta, \varphi)$ to get a new coloring φ'''. Thus $\gamma \in \overline{\varphi}'''(w(e_{l_s})) \cap \overline{\varphi}'''(u) \cap \overline{\varphi}'''(v)$, as desired because γ is the desired color instead of δ. Now we assume that $\delta \in \overline{\varphi}(w(e_{l_s}))$ and $\delta \neq \beta_{l_1}$. Then $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi) = P_v(\delta, \beta_{l_s}, \varphi)$ by the observation II. Note that e_{l_s} is on $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi)$. Let the coloring $\varphi_1 = \varphi / P_u(\delta, \beta_{l_s}, \varphi)$. Hence S_y is still a y-generated ce-sequence under φ_1 with $\beta_{l_s} \in \overline{\varphi}_1(u)$.

Next we show that we may assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. If $w(e_{l_s}) = x$, we are done. Otherwise, $w(e_{l_s}) = y$. We have $P_x(\eta, \delta, \varphi_1) = P_y(\eta, \delta, \varphi_1)$ by Lemma 3.1. The coloring $\varphi_1' = \varphi_1 / P_x(\eta, \delta, \varphi)$ results in $\eta \in \overline{\varphi}_1'(y)$, as desired. Now note that $P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1) = P_u(\eta, \beta_{l_s}, \varphi_1) = w(e_{l_s}) z_{l_s} u$. Then the coloring $\varphi_2 = \varphi_1 / P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1)$ results in $\beta_{l_s} \in \overline{\varphi}_2(w(e_{l_s})) \cap (\varphi_2)^s_{w(e_{l_{s-1}})}(z_{l_{s-1}})$, contradicting the minimality of s. This completes the proof of Case 2.

Combining the above Cases 1 and 2, we complete the proof of Claim 1 for $s \geq 2$. Together with the proof of Claim 1 for $s \leq 1$, we prove Claim 1. □

Claim 2. The union of vertex sets of any two linear ce-sequences is φ^{ce}-elementary.

Proof. Suppose that Claim 2 is false. Without loss of generality, we choose φ such that there exist two linear ce-sequences $S_1 = (x, e, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ and $S_2 = (x, e, y, e_{l_1'}, z_{l_1'}, \ldots, e_{l_t'}, z_{l_t'})$ whose union of vertex sets is not φ^{ce}-elementary with $s + t$ as small as possible, where $s, t \geq 1$. Note that $V(S_1)$ and $V(S_2)$ are φ^{ce}-elementary by Claim 1. By the minimality of $s + t$, $z_{l_s} \neq z_{l_t'}$ and there exists a color $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap C_{\varphi, w(e_{l_t'})}(z_{l_t'})$. We consider the following three cases. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l_t'})$, then z_{l_s} and $z_{l_t'}$ are respectively not repeated vertices in S_1 and S_2 since the minimality of $s+t$. By the same proof of Claim 2 in Theorem 1.3, we can obtain three endvertices on one Kempe chain, which gives a contradiction.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(z_{l_t'})$ (or $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_{w(e_{l_t'})}(z_{l_t'})$ by symmetry), then there is an edge $e' = z_{l_s} u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$. By the definition of linear ce-sequence in C-e-fan and the minimality of $s + t$, z_{l_s} may be a repeated vertex in S_1, while $z_{l_t'}$ is not a repeated vertex in S_2. Note that $\varphi(e_{l_1})$ and $\varphi(e_{l_1'})$ are in $C_{\varphi, y}(x) \cup C_{\varphi, x}(y)$. ($\varphi(e_{l_1})$ and $\varphi(e_{l_1'})$ could be the same color.) We consider the following two subcases. If $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l_1'})\}$, then we have $P_{w(e_{l_s})}(\delta, \eta, \varphi) = P_u(\delta, \eta, \varphi)$ by the observation II since S_1 is φ^{ce}-elementary. Similarly, we have $P_{w(e_{l_s})}(\delta, \eta, \varphi) = P_{z_{l_t'}}(\delta, \eta, \varphi)$ by the observation II since S_2 is φ^{ce}-elementary. Thus $w(e_{l_s}), z_{l_t'}$ and u are three endvertices of $P_{w(e_{l_s})}(\delta, \eta, \varphi)$, which gives a contradiction. Now we consider the remaining case $\delta \in \{\varphi(e_{l_1}), \varphi(e_{l_1'})\}$. Let $w'(e_{l_s}) \in \{x, y\} \backslash \{w(e_{l_s})\}$. Recall that $max\{d(x), d(y)\} \leq \Delta - 1$. Hence we can choose a color $\gamma \in \overline{\varphi}(w'(e_{l_s}))$ with $\gamma \notin \{\varphi(e_{l_1}), \varphi(e_{l_1'})\}$. We have $P_x(\delta, \gamma, \varphi) = P_y(\delta, \gamma, \varphi)$ by Lemma 3.1. Do Kempe change on $P_u(\delta, \gamma, \varphi)$ to get a new coloring φ_1. Thus $\gamma \in \overline{\varphi}_1(w'(e_{l_s})) \cap \overline{\varphi}_1(u)$.

Similarly as the subcase above (when $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$), we have $P_{w'(e_{l_s})}(\gamma, \eta, \varphi_1) = P_{z_{l'_t}}(\gamma, \eta, \varphi_1)$ and $P_{w'(e_{l_s})}(\gamma, \eta, \varphi_1) = P_u(\gamma, \eta, \varphi_1)$. Thus $w'(e_{l_s}), z_{l'_t}$ and u are three endvertices of $P_{w'(e_{l_s})}(\delta, \eta, \varphi_1)$, which also gives a contradiction.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_{w(e_{l'_t})}(z_{l'_t})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l'_t}v$ such that $v \neq w(e_{l'_t})$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(w(e_{l'_t}))}{2}$. Obviously, $u \neq v$, and z_{l_s} and $z_{l'_t}$ may be repeated vertices respectively in S_1 and S_2. Without loss of generality, we suppose that $d(x) \leq d(y)$. It follows from Lemma 3.4 that there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We consider the following two subcases. If $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$, then we have $P_x(\delta, \eta, \varphi) = P_u(\delta, \eta, \varphi)$ by the observation II. Similarly, we have $P_x(\delta, \eta, \varphi) = P_v(\delta, \eta, \varphi)$. Thus x, u and v are three endvertices on one (δ, η)-chain, which is a contradiction. Now we consider the remaining case $\delta \in \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. Recall that $max\{d(x), d(y)\} \leq \Delta - 1$. Hence we can choose a color $\gamma \in \overline{\varphi}(y)$ with $\gamma \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. We have $P_x(\delta, \gamma, \varphi) = P_y(\delta, \gamma, \varphi)$ by Lemma 3.1. Do Kempe changes on $P_u(\delta, \gamma, \varphi)$ and $P_v(\delta, \gamma, \varphi)$ to get a new coloring φ_1. Thus we have $\gamma \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(u) \cap \overline{\varphi}_1(v)$. Thus we are back to the previous subcase with y in place of x and γ in place of δ. This completes the proof of Claim 2. □

Now we are ready to show that $V(F^{ce})$ is φ^{ce}-elementary. Suppose not. Note that $\{x, y\}$ is φ^{ce}-elementary and each linear ce-sequence in F^{ce} contains vertices x and y. There exist one color η and two distinct vertices z_i and z_j such that $\eta \in C_{\varphi, w(e_i)}(z_i) \cap C_{\varphi, w(e_j)}(z_j)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$. By the definition of simple C-e-fan, there exist two linear ce-sequences with z_i and z_j respectively as the last vertex, which is a contradiction to Claim 1 for $i = 0$ or a contradiction to Claim 2 for $1 \leq i \leq p - 1$. This proves that $V(F^{ce})$ is φ^{ce}-elementary.

Now we show the "furthermore" part. We assume that F^{ce} is maximal. Let the edge set $\Gamma = \{e_1, \ldots, e_p\}$ and the color set $\Gamma' = \bigcup_{z \in V(F^{ce})} C_\varphi(z)$. Note that $C_\varphi(x)$, $C_\varphi(y)$ and $C_\varphi(z)$, where $z \in V(F^{ce}) \backslash \{x, y\}$, are disjoint since $V(F^{ce})$ is φ^{ce}-elementary. We have

$$p = |\Gamma| = \sum_{z \in V(F^{ce}) \backslash \{x,y\}} (\mu_{F^{ce}}(x, z) + \mu_{F^{ce}}(y, z)) = |\Gamma^*|. \tag{3}$$

Now we calculate $|\Gamma^*|$ in another way. By the definition of C-e-fan, $\varphi(e_i) \in \Gamma'$ for each $i \in [p]$. By the maximality of F^{ce}, for any $\alpha \in \Gamma'$, α appears exactly once in Γ^* if $\alpha \in C_\varphi(x) \cup C_\varphi(y)$. Otherwise, α appears exactly twice in Γ^*. Thus we have

$$|\Gamma^*| = |C_\varphi(x)| + |C_\varphi(y)| + \sum_{z \in V(F^{ce}) \backslash \{x,y\}} 2|C_\varphi(z)|. \tag{4}$$

Combining equations (3) and (4), we prove that

$$|C_\varphi(x)| + |C_\varphi(y)| = \sum_{z \in V(F^{ce}) \backslash \{x,y\}} (\mu_{F^{ce}}(x, z) + \mu_{F^{ce}}(y, z) - 2|C_\varphi(z)|).$$

□

Remark: For Theorem 2.3, the condition **C3** is used in the proof of Claim 1 in Case 2. We believe this condition is necessary but do not have examples to support this claim.